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A New Idea for Realizing Positive Real Immittance Functions of Even Rank

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Abstract

A novel electrical circuit designed according to the theory postulated in this paper delivers a prescribed immittance response. A driving-point function $\bar{F}(s)$ is realized by a single-T circuit and/or a single-π circuit, implying a negative resistor but otherwise positive elements; a positive real immittance function F(s) is realized by the same circuits augmented by an immittance kf(s) of low rank at the input.

The circuit has high reliability and a rugged stability in the face of environmental changes. In addition, its low weight and small size fulfill requirements for military applications.

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A New Idea for Realizing Positive Real Immittance Functions of Even Rank

1. INTRODUCTION

The positive real (PR) function originally defined by Otto Brune (1931) represents a general class of rational functions composed of several subclasses. The subclass treated in this paper,

$$F(s) = \frac{N(s)}{D(s)} = \frac{s^{\nu} + N_{\nu-1}s^{\nu-1} + \ldots + N_1 s + N_0}{s^{\nu} + D_{\nu-1}s^{\nu-1} + \ldots + D_1 s + D_0}\,, \tag{1}$$

is a function of even rank (2ν), and denoted as a function of the ER class. All its coefficients N_i and D_i are nonnegative. For convenience we assume that the polynomials N(s) and D(s) are normalized* by $N_{\nu} = D_{\nu} = +1$.

The function

$$\overline{F}(s) = \frac{\overline{N}(s)}{\overline{D}(s)} = \frac{s^{\nu} + \overline{N}_{\nu-1}s^{\nu-1} + \ldots + \overline{N}_1 s + \overline{N}_0}{s^{\nu} + \overline{D}_{\nu-1}s^{\nu-1} + \ldots + \overline{D}_1 s + \overline{D}_0}\,, \tag{2}$$

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*Unless otherwise stated, the term polynomial as used in this paper always includes normalization in this sense.

considered as belonging to a subclass of the ER class, is denoted as a function of the H class. This function $\bar{F}(s)$, treated in Sec. 1, has some properties that F(s) does not have.

We will show that the function $\bar{F}(s)$ can be realized by the circuits in Figure 1,* where it is the driving-point immittance. The terminating immittances Z(s) and the immittance branches X(s) are positive. One of the resistances (conductances) R_u, R_v, R_w is negative; it has to be realized by any suitable active device. If the degree ν is an even one, then X(s) and Z(s) are PR functions of rank $\nu-1$; if the degree ν is odd, then one immittance is of rank ν and the other is of rank $\nu-2$. We shall later assume that $\bar{F}(s)$ is known by its coefficients, and design the circuit.

In Sec. 2 we will show that it is possible to derive an H-class function $\bar{F}(s)$ from an ER-class function F(s) of the same rank by adding an impedance of low rank in series or in shunt to the circuit that realizes $\bar{F}(s)$. For any degree ν the circuit realizing F(s) must be a single T or single π, terminated at both ends.

2. REALIZATION AND DEFINITION OF AN H-CLASS FUNCTION $\bar{F}(s)$

2.1 The Norm Function $\bar{F}(s)$ and Its Realization

Consider the circuits in Figure 2. With $\bar{n}$ a positive constant and V a positive immittance, let

$$U = V(\bar{n}-1) \tag{3a}$$

and

$$W = -V(\bar{n}-1)/\bar{n} \, . \tag{3b}$$

Then

$$1/U + 1/V + 1/W = UV + VW + UW = 0 \, , \tag{4}$$

$$U + V = \bar{n}V \, , \tag{5a}$$

$$V + W = V/\bar{n} \, , \tag{5b}$$

$$U + W = V(\bar{n}-1)^2/\bar{n} \, . \tag{5c}$$

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* In this figure, as in all others in this report, the terminated-T circuit (a) implies impedance branches, and the terminated-π circuit (b) implies admittance branches.

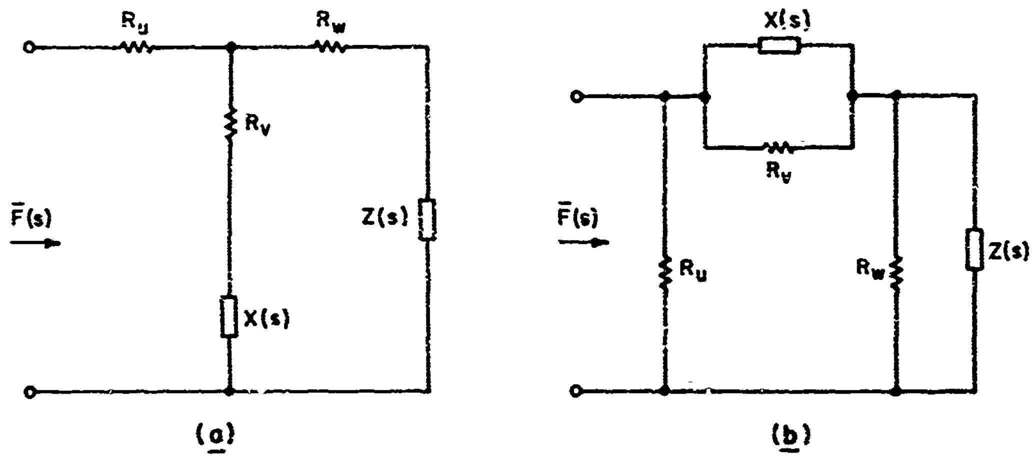

Figure 1. Terminated-T Circuit (a) and Terminated-π Circuit (b) Where $\bar{F}(s)$ is the Driving-point Immittance

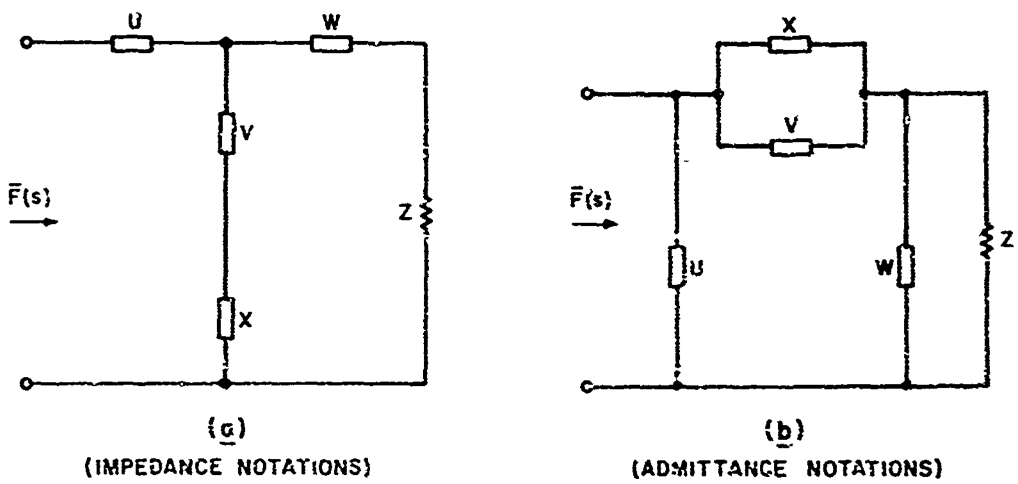

Figure 2. Notation (a) for Terminated-T Circuit (Impedance Branches) and (b) for Terminated-π Circuit (Admittance Branches)

The driving-point immittance of each of the circuits in Figure 2 is

$$\bar{F}(s) = \frac{Z(U+V)+X(U+W)+XZ}{(V+W)+X+Z} = \frac{VZ\bar{n}^2+VX(\bar{n}-1)^2+XZ\bar{n}}{V+X\bar{n}+Z\bar{n}} . \tag{6}$$

With positive constants v, x, and z, normalized PR functions $\phi(s)$ and $\Phi(s)$, and $\xi(s) = 1$ (dummy function), we introduce

$$V = v\phi(s) , \tag{7a}$$

$$X = x\Phi(s) , \tag{7b}$$

$$Z = z\xi(s) = z , \tag{7c}$$

and set

$$z = 1/\bar{n}^2 . \tag{7d}$$

Then by substitution in Eq. (6),

$$\bar{F}(s) = \frac{\phi(s)+x(\bar{n}-1)^2\phi(s)\Phi(s)+x\Phi(s)/v\bar{n}}{\phi(s)+1/v\bar{n}+x\bar{n}\Phi(s)/v} . \tag{8}$$

We denote the form of the function presented in Eq. (8) as the <u>norm.</u> Its realization according to Figure 2 needs an inverter circuit for U (which is negative if $\bar{n} < 1$) or for W (which is negative if $\bar{n} > 1$). In both events the inverter presents the technical disadvantage of being frequency-dependent since U and W imply $\phi(s)$.

2.2 Immittance Functions $\bar{F}_A(s)$ and $\bar{F}_B(s)$ to be Realized

In an earlier paper (1966) we showed that $\bar{F}(s)$ remains invariant when in any pair of the functions $\phi(s)$, $\Phi(s)$, and $\xi(s)$, the functions are interchanged, or when the functions are replaced by their inverses and then interchanged, provided that the original constants $\bar{n}$, v, x are at the same time transformed to new constants $\bar{n}'$, v', and x'. For our present purpose we interchange $\phi(s)$ and $\xi(s)$ and obtain $\bar{F}_A(s) = \bar{F}(s)$. We then interchange the inverses of these functions and obtain $\bar{F}_B(s) = 1/\bar{F}(s)$. The transformation formulas are listed in Table 1. What we have achieved is that in $\bar{F}_A(s)$ and $\bar{F}_B(s)$, V' = v'; concomitantly, U' and W' become mere constants and thus independent of frequency. But $\phi(s)$ and $\Phi(s)$ become associated with the constants z' and x', which are certainly positive. Equivalent block circuits realizing $\bar{F}(s) = \bar{F}_A(s)$ and $\bar{F}(s) = \bar{F}_B(s)$ are presented in Figure 3.

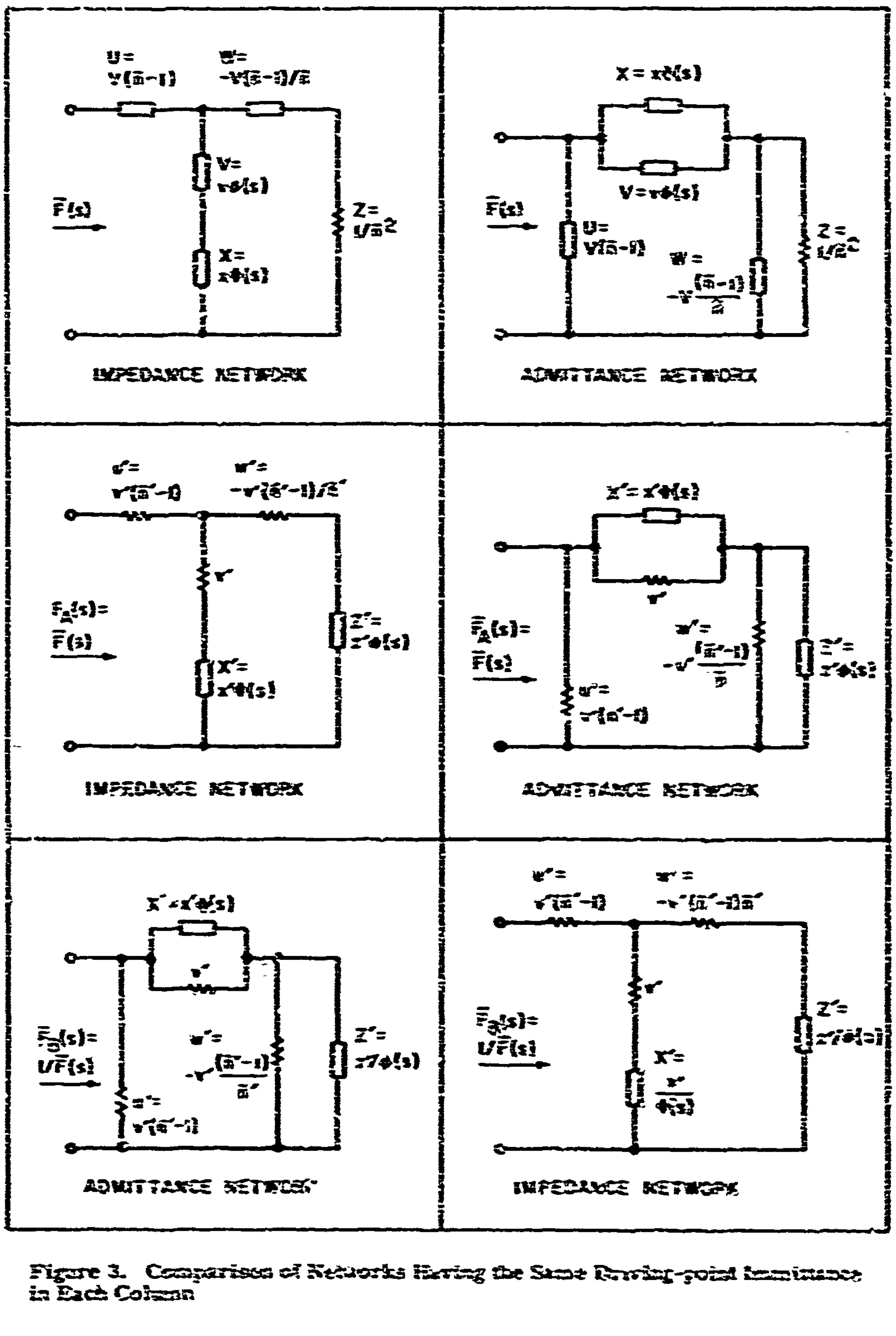

Figure 3. Comparison of Networks Having the Same Driving-point Immittance in Each Column

Table 1. Transformations of the Constants of $\bar{F}(s)$ for $\bar{F}_A(s)$ $(\phi \longleftrightarrow \xi,\ \Phi)$ and $1/\bar{F}_B(s)$ $(1/\phi \longleftrightarrow 1/\xi,\ 1/\Phi)$

	$\bar{F}_A(s) = \bar{F}(s)$	$1/\bar{F}_B(s) = \bar{F}(s)$
$\bar{n}'$	$\bar{n}/(\bar{n}-1)$	$\bar{n}/(\bar{n}-1)$
u'	$1/\bar{n}$	$\bar{n}$
v'	$(\bar{n}-1)/\bar{n}$	$\bar{n}(\bar{n}-1)$
w'	$(1-\bar{n})/\bar{n}^2$	$1-\bar{n}$
x'	$x(\bar{n}-1)/\bar{n}^2$	$1/x$
z'	$v(\bar{n}-1)^2/\bar{n}$	$1/v\bar{n}$

2.3 The PR and Normalized Functions $\phi(s)$ and $\Phi(s)$

Let a(s) and b(s) be two polynomials of degree ν_a, and let $\alpha(s)$ and $\beta(s)$ be two polynomials of degree ν_α. Assume that

$$\phi(s) = s\,\frac{a(s)}{b(s)} \text{ is PR} \tag{9}$$

and

$$\Phi(s) = \frac{\beta(s)}{s\alpha(s)} \text{ is PR .} \tag{10}$$

Substituting these functions in Eq. (8) yields

$$\bar{F}(s) = \frac{s^2 a(s)\alpha(s) + x(\bar{n}-1)^2 s a(s)\beta(s) + x b(s)\beta(s)/v\bar{n}}{s^2 a(s)\alpha(s) + s\alpha(s)b(s)/v\bar{n} + x\bar{n} b(s)\beta(s)/v} \;. \tag{11}$$

The polynomials in the numerator and denominator of Eq. (11) are of degree $2 + \nu_a + \nu_\alpha$ and normalized. Their degrees are the same as those of $\bar{N}(s)$ and $\bar{D}(s)$ in Eq. (2) when we provide that

$$\text{for even degree } \nu, \quad \nu_a = \nu_\alpha = \frac{1}{2}(\nu - 1)\;; \tag{12}$$

$$\text{for odd degree } \nu, \quad \nu_a = \frac{1}{2}(\nu - 1), \tag{13a}$$

$$\nu_\alpha = \frac{1}{2}(\nu - 3)\,. \tag{13b}$$

For example,

$$\text{if } \nu = 2, \quad \phi(s) = 1/\Phi(s) = s \; ; \tag{14}$$

$$\text{if } \nu = 3, \quad \phi(s) = s(s+a_o)/(s+b_o), \tag{15a}$$

$$\Phi(s) = 1/s \; ; \tag{15b}$$

$$\text{if } \nu = 4, \quad \phi(s) = s(s+a_o)/(s+b_o), \tag{16a}$$

$$\Phi(s) = (s+\alpha_o)/s(s+\beta_o). \tag{16b}$$

The results of a comparison of the coefficients in the expressions for $\bar{F}(s)$ given in Eqs. (2) and (11) are listed in Table 2, from which impedance and admittance realizations of $\phi(s)$ and $\Phi(s)$ can easily be found.

Table 2. Coefficients of $\bar{N}(s)$ and $\bar{D}(s)$ for Ranks 2, 3, and 4

$\bar{N}_i$	$\nu =$		
	2	3	4
$\bar{N}_o$	$\bar{N}_B$	$b_o\bar{N}_B$	$b_o\beta_c\bar{N}_B$
$\bar{N}_1$	$\bar{N}_A$	$a_o\bar{N}_A+\bar{N}_B$	$a_o\beta_o\bar{N}_A + (b_o+\beta_o)\bar{N}_B$
$\bar{N}_2$	1	a_o+N_A	$a_o\alpha_o + (a_o+\beta_o)\bar{N}_A + \bar{N}_B$
$\bar{N}_3$	-	1	$a_o+\alpha_o+\bar{N}_A$
$\bar{N}_4$	-	-	1
$\bar{D}_i$			
$\bar{D}_o$	$\bar{D}_b$	$b_o\bar{D}_B$	$b_o\beta_o\bar{D}_B$
$\bar{D}_1$	$\bar{D}_a$	$b_o\bar{D}_A+\bar{D}_B$	$\alpha_o b_o\bar{D}_A + (b_o+\beta_o)\bar{D}_B$
$\bar{D}_2$	1	$a_o+\bar{D}_A$	$a_o\alpha_o + (\alpha_o+b_o)\bar{D}_A + \bar{D}_B$
$\bar{D}_3$	-	1	$a_o+\alpha_o+\bar{D}_A$
$\bar{D}_4$	-	-	1
	$\bar{N}_A = x(\bar{n}-1)^2$, $\quad \bar{N}_B = x/v\bar{n}$; $\bar{D}_A = 1/v\bar{n}$, $\quad \bar{D}_B = x\bar{n}/v$.		

2.4 Derivation of the Constants $\bar{n}$, v, x, and of the Functions $\phi(s)$ and $\Phi(s)$ From the Coefficients of $\bar{N}(s)$ and $\bar{D}(s)$ of $\bar{F}(s)$

Assume now that we know the 2ν coefficients $\bar{N}_o, \ldots, \bar{N}_{\nu-1}$, and $\bar{D}_o, \ldots, \bar{D}_{\nu-1}$. We want to find the three constants $\bar{n}$, v, x, and the $\nu_a + \nu_\alpha$ coefficients of $\phi(s)$ and $\Phi(s)$, a total of $3+\nu_a+\nu_\alpha$ unknowns. Table 2 offers $4+\nu_a+\nu_\alpha$ meaningful equations, one more than we need. This surplus equation holds only when the system of equations is consistent, which is the case if (a) the proposed coefficients are actually those of an H-class function $\bar{F}(s)$, and (b) if our assumptions are correct. [We shall test both conditions later.] Unfortunately, as a glance at $\nu = 4$ shows, the equations in Table 2 are nonlinear. The more ν increases, the more complicated they become. To be able to cope with any degree ν, we have to find another way of determining the unknowns.

Note that for any degree ν,

$$\bar{N}_o = x b_o \beta_o / v\bar{n} , \tag{17a}$$

$$\bar{D}_o = x\bar{n} b_o \beta_o / v . \tag{17b}$$

Therefore,

$$\bar{n} = + \sqrt{\bar{D}_o / \bar{N}_o} . \tag{18}$$

Thus, the constant $\bar{n}$ is immediately known.

Let us define the polynomials:

$$P(s) = sa(s) - b(s)/v\bar{n}(\bar{n}-1) , \tag{19}$$

$$Q(s) = s\alpha(s) + x\bar{n}(\bar{n}-1)\beta(s) , \tag{20}$$

$$\bar{d}(s) = [\bar{N}(s) - \bar{D}(s)]/(\bar{n}-1) . \tag{21}$$

The polynomials P(s) and Q(s) are normalized, but $\bar{d}(s)$ is not. The degrees ν_P and ν_Q are defined as follows:

$$\text{If } \nu \text{ is even, } \nu_P = \nu_Q = \frac{1}{2}\nu . \tag{22}$$

$$\text{If } \nu \text{ is odd, } \nu_P = \frac{1}{2}(\nu+1); \ \nu_Q = \frac{1}{2}(\nu-1) . \tag{23}$$

$$\text{The degree of } \bar{d}(s) \text{ is } \nu-1 . \tag{24}$$

Since the coefficients of $\phi(s)$ and $\Phi(s)$ are positive,

$$P(s) \text{ has only positive coefficients if } \bar{n} < 1 , \tag{25}$$

$$Q(s) \text{ has only positive coefficients if } \bar{n} > 1 . \tag{26}$$

From the identity of Eqs. (11) and (2) it follows that

$$\begin{aligned} P(s)Q(s) &= [\, \bar{n}\bar{N}(s) - \bar{D}(s)] / (\bar{n}-1) \\ &= s^{\nu} + (PQ)_{\nu-1}s^{\nu-1} + \ldots + (PQ)_1 s + P_oQ_o . \end{aligned} \tag{27}$$

The polynomial product P(s)Q(s) is known in its summation form. By solving the equation P(s)Q(s) = 0, we can transform it into the product form. In this form it consists of the product of some linear polynomial factors (corresponding to the real roots) and some quadratic polynomial factors, each of the latter having a negative discriminant (corresponding to the conjugate complex roots). From this product we are able to attribute some factors to P(s) and others to Q(s) according to the information given by Eqs. (22), (23), (25), and (26). But some ambiguity may remain since some factors may be either in P(s) or in Q(s). As we shall see later, this ambiguity can be cleared by using the surplus equation.

As an example, assume that for a biquartic function $\bar{F}(s)$ (ν=4), with $\bar{n} > 1$,

$$\begin{aligned} P(s)Q(s) &= s^4 + (PQ)_3 s^3 + (PQ)_2 s^2 + (PQ)_1 s + P_oQ_o \\ &= (s-s_4)(s+s_1)(s+s_2)(s+s_3) , \end{aligned} \tag{28}$$

where s_4, s_1, s_2, and s_3 are positive, and $s_1 < s_2 < s_3$. It is clear that $(s-s_4)$ is a factor in P(s) only when $\bar{n} > 1$. One of the other three factors is also in P(s). We thus have three choices for distributing these factors over P(s) and Q(s), as shown in Table 3.

Table 3

Choice	P_o	P_1	Q_o	Q_1
No. 1	$-s_4s_1$	s_1-s_4	s_2s_3	s_2+s_3
No. 2	$-s_4s_2$	s_2-s_4	s_1s_3	s_1+s_3
No. 3	$-s_4s_3$	s_3-s_4	s_1s_2	s_1+s_2

As another example let us consider a bicubic function $\overline{F}(s)$ ($\nu=3$) for which we may have found that

$$P(s)Q(s) = s^3 + (PQ)_2 s^2 + (PQ)_1 s + P_o Q_o$$

$$= (s-s_3)(s+s_1)(s+s_2) , \qquad (29)$$

where s_3, s_1, and s_2 are positive, and $s_1 < s_2$. In this example, if $\overline{n} < 1$, then $(s-s_3)$ is a factor in $Q(s)$ and the other two factors are in $P(s)$. But if $\overline{n} > 1$, then we are left with the two choices in Table 4.

Table 4

Choice	P_o	P_1	Q_o
No. 1	$-s_1s_3$	s_3-s_1	s_2
No. 2	$-s_2s_3$	s_2-s_3	s_1

Before going on to clear these ambiguities, we find by some trivial algebraic operations that

$$\bar{d}(s) = (\bar{n}+1)P(s)Q(s)/\bar{n} - s[\alpha(s)P(s)/\bar{n} + a(s)Q(s)] ,$$

and by ordering this equation we get

$$a(s)Q(s) = -\alpha(s)P(s)/\bar{n} + (\bar{n}+1)P(s)Q(s)/[s\bar{n} - \bar{d}(s)]/s . \qquad (30)$$

The only unknowns in Eq. (30) are the positive coefficients of $a(s)$ and $\alpha(s)$. On both sides of this equation there are normalized polynomials of degree $\nu-1$. Comparing coefficients yields $\nu-1$ meaningful equations by which we are able to determine the $\nu-2$ unknowns. Here we also have a surplus equation. In contrast to the results of the earlier coefficient comparison, however, the equations derived from Eq. (30) are linear and can be solved by applying Cramer's rule. On the right side of Eq. (30) we have a comparison in s^{-1} that has no match on the left side. It yields the triviality

$$(\bar{n}+1)P_oQ_o - \bar{n}\bar{d}_o = 0 . \qquad (31)$$

When all the unknown coefficients of a(s) and α(s) have been determined, we find the remaining unknowns v and x by

$$1/v\bar{n}(\bar{n}-1) = a_{\nu_a - 1} - P_{\nu_a} , \tag{32}$$

$$x\bar{n}(\bar{n}-1) = Q_{\nu_\alpha} - \alpha_{\nu_\alpha - 1} . \tag{33}$$

The coefficients of b(s) and β(s) can be obtained by comparing coefficients in Eqs. (19) and (20).

2.5 Instructions for Solving the System of Linear Equations

We assume familiarity with Cramer's rule (see Hildebrand, 1956, among others). The derivation of the matrix system to which the rule is applied deserves some discussion. These matrixes can be written almost immediately since their elements have to be taken from the coefficients of P(s) and Q(s) for each of the possible ambiguous choices.

Each equation derived from (30) is a comparison of coefficients associated with s^i and has the form

$$\sum_{j=0}^{\nu_a} a_{i-j}Q_j + \sum_{k=0}^{\nu_a} \alpha_{i-k}P_k/\bar{n} = (\bar{n}+1)(PQ)_{i+1}/\bar{n} - \bar{d}_{i+1} , \tag{34}$$

where, by definition,

$$a_{i-j} = 0 \text{ for } i-j > \nu_a , \tag{34a}$$

$$\alpha_{i-k} = 0 \text{ for } i-k > \nu_\alpha , \tag{34b}$$

$$P_k = 0 \text{ for } k < 0 , \tag{34c}$$

$$Q_j = 0 \text{ for } j < 0 . \tag{34d}$$

Suppose we write these equations in sequence, starting with $i = \nu-2$ at the top and ending with $i = 1$ at the bottom (we need only these equations to determine the $\nu - 2$ unknowns). Because of the normalizations there are some constant terms that we must transpose to the right side, obtaining

$$\sum_{j=0}^{\nu_a - 1} a_{i-j} Q_j + \sum_{k=0}^{\nu_\alpha - 1} \alpha_{i-k} P_k / \bar{n} = C_i = A_{i+1} - \bar{d}_{i+1} \, , \tag{35}$$

where

$$\bar{n} A_{i+1} = (\bar{n}+1)(PQ)_{i+1} - (P_{i+1-\nu_\alpha} + \bar{n} Q_{i+1-\nu_a}) \, ; \tag{35a}$$

$$\bar{d}_{i+1} = (\bar{N}_{i+1} - \bar{D}_{i+1}) / (\bar{n} - 1) \, . \tag{35b}$$

From the left side of these equations we derive a square matrix $\| Q_{i,j}, P_{i,k} \|$ of ν-2 rows and columns. This matrix has a group of columns listing the coefficients of Q(s) and another group listing the coefficients of P(s). The first column in each group lists the coefficients sequentially downward, starting at the top with $Q_{\nu_Q} = 1$ and $P_{\nu_P} = 1$. The list is repeated from one column to the next but shifted downward by one position. The places thus vacated then contain zero elements.

From the right side of the equations we derive a column matrix $\| C_i \|$ of ν-2 rows. The difference between two column matrixes, this matrix is expressed by

$$\| C_i \| = \| A_{i+1} \| - \| \bar{d}_{i+1} \| \, . \tag{36}$$

With these column matrixes known, we can apply Cramer's rule to get:

$$\Delta \;\; = \det | Q_{i,j}, P_{i,k} | \;\; ; \tag{37a}$$

$$\Delta_{a_j} = \det | Q_{i,j}, P_{i,k} | \;\; , \tag{37b}$$

with column j replaced by (36);

$$\Delta_{\alpha_k} = \det | Q_{i,j}, P_{i,k} | \;\; , \tag{37c}$$

with column k also replaced by (36) .

Then

$$a_j = \Delta_{a_j} / \Delta \, ; \tag{38a}$$

and

$$\alpha_k / \bar{n} \;\; = \Delta_{\alpha_k} / \Delta \, , \tag{38b}$$

2.6 The Equation of Consistency and the Definition of H-Class Functions $\overline{F}(s)$

As used here, the surplus equation in our system is the one that compared the coefficients associated with s^0. When we substitute the coefficients obtained through Eqs. (38) we obtain

$$\sum_{i=1}^{\nu-1} A_i c_i - \sum_{i=1}^{\nu-1} \overline{d}_i c_i = 0 , \tag{39}$$

for which A_i and $\overline{d}_i$ are given by Eqs. (35), and c_i are the determinants multiplied with cofactors of a square matrix that is obtained when the square matrix of $\nu-2$ rows and columns is extended by the row $i=0$ and one row is deleted. Assume, for example, the degree $\nu=6$. The extended matrix in this case is then

$$\begin{array}{c|cccc} i \backslash j & 1 & 0 & 1 & 0 \; k \\ \hline 4 & & & & \\ 3 & & & & \\ 2 & & & & \\ 1 & & & & \\ 0 & & & & \end{array}
\left\Vert \begin{array}{cccc} Q_3=1 & 0 & P_3=1 & 0 \\ Q_2 & 1 & P_2 & 1 \\ Q_1 & Q_2 & P_1 & P_3 \\ Q_o & Q_1 & P_o & P_2 \\ 0 & Q_o & 0 & P_1 \end{array} \right\Vert . \tag{40}$$

The coefficients $+c_1$, $-c_2$, $+c_3$, $-c_4$, and $+c_5$ are obtained by respectively deleting the consecutive rows 0, 1, 2, 3, and 4. The coefficient c_o is defined as

$$c_o = -\sum_{i=1}^{\nu-1} A_i c_i / A_o , \tag{41}$$

with $A_o = (\overline{n}+1)P_o Q_o/\overline{n}$ according to (35a). By (31) we can add $(A_o - \overline{d}_o)c_o$ on the left side in (41) and obtain

$$\delta = -\sum_{i=0}^{\nu-1} \overline{d}_i c_i = 0 \text{ as the } \underline{\text{Equation of Consistency.}} \tag{42}$$

Equation (42) yields $\delta = 0$ only if (a) the proposed coefficients $\overline{N}_i$ and $\overline{D}_i$ are those of an H-class function $\overline{F}(s)$, and (b) the correct choice has been made in selecting the factors of P(s) and Q(s) from the product P(s)Q(s) in Eq. (27). Thus, this equation also defines the H-class function.

Compact formulas for the coefficients of c_i can be given for the relatively low degrees $\nu = 3$ and $\nu = 4$.

For $\nu = 3$,

$$c_2 = -Q_o, \quad c_1 = 1, \quad c_o = -[\bar{n}P_o + Q_o(P_1 - Q_o)]/(\bar{n}+1)P_oQ_o . \tag{43}$$

For $\nu = 4$,

$$c_3 = P_1Q_o - P_oQ_1, \quad c_2 = P_o - Q_o, \quad c_1 = Q_1 - P_1 ,$$

$$c_o = -[\bar{n}P_o(c_2 + c_1Q_o) + Q_o(c_2 + c_1P_1)]/(\bar{n}+1)P_oQ_o . \tag{44}$$

2.7 Numerical Example No. 1 (Biquartic II-Class Function)

Let the function $\bar{F}(s)$ of degree $\nu = 4$ have the coefficients listed in Table 5, from which we compute the coefficients of $\bar{d}(s)$ by Eq. (21). With $\bar{n} = 2.0$, we use Eq. (18) to obtain

$$P(s)Q(s) = s^4 + 15s^3 + 63s^2 + 41s - 120$$
$$= (s-1)(s+3)(s+5)(s+8) . \tag{45}$$

Table 5

i	$\bar{N}_i$	$\bar{D}_i$	$\bar{d}_i$
0	60	240	-180
1	118	195	-77
2	75.5	88	-12.5
3	15.5	16	-0.5
4	1.0	1.0	0

The three choices for selecting P(s) and Q(s) from this product are listed in Table 6 together with the coefficients c_i computed by (44) and δ by (42) for each of the choices. According to the last column in Table 6, the correct choice is No. 2, for which by Eqs. (37) we obtain

$$\Delta = -7, \quad \Delta_{a_o} = -45.5, \quad \Delta_{\alpha_o} = -49 . \tag{46}$$

Table 6

Choice	P_0	P_1	Q_0	Q_1	c_3	c_2	c_1	c_o	δ
No. 1	-3	2	40	13	119	-43	11	-4	-351
No. 2	-5	4	24	11	151	-29	7	-1.4	0
No. 3	-8	7	15	8	169	-23	1	0	-126

Then, by Eqs. (38), we find

$$a_o = 6.5, \quad \alpha_o = 7 ; \tag{46a}$$

and by (32) and (33),

$$v = 0.2, \quad x = 2. \tag{46b}$$

Coefficient comparison in (19) and (20) yields

$$b_o = 2, \quad \beta_o = 6. \tag{46c}$$

The two equivalent circuits realizing the impedance $\bar{F}(s)$ are shown in Figure 4. Their elements computed according to Tables 1 and Figure 3 are listed in Table 7.

Table 7. Circuit Elements in Figure 4 for Examples Nos. 1 and 2

	Example No. 1		Example No. 2	
	$\bar{F}_A(s) = \bar{F}(s)$	$1/\bar{F}_B(s) = \bar{F}(s)$	$\bar{F}_A(s) = \bar{F}(s)$	$1/F_B(s) = F(s)$
R_1	0.5	0.5	0.2745587	0.2745587
R_2	0.5	0.5	0.7254412	0.1039126
R_3	-0.25	-1.0	-0.1991762	-0.3784714
R_4	2.0	2.0	2.3095979	0.3308403
R_5	169/189	338/90	1.6368449	3.1103057
L_1	0.325	1.3	0.3733889	0.7095074
L_2	13/90	52/90	0.2708061	0.5145811
C_1	0.5	0.5	0.3654515	2.5513119
C_2	0.5/6	0.5/6	0.0799116	0.5578634

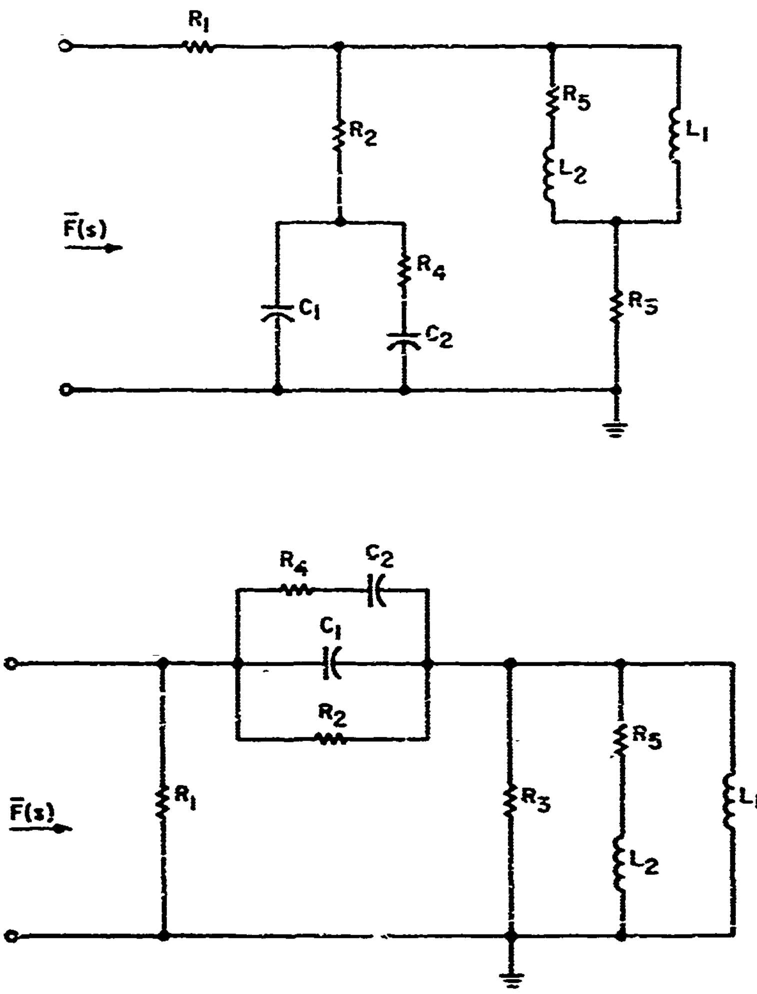

Figure 4. Two Equivalent Circuits Realizing the Driving-point Impedance $\bar{F}(s)$ [$\nu = 4$; grounded negative resistance R_3]

3. Realization of ER-Class Functions $\bar{F}(s)$

If a PR function F(s) that is of even rank and has known coefficients is treated as discussed in Sec. 2 and Eq. (42) does not yield $\delta = 0$ for any of the possible choices, then the function F(s) is not an H-class function. An H-class function can, however, be obtained by splitting a suitable PR function kf(s) from F(s).

Two splits, yielding dual and equivalent circuits, are feasible:

$$K\bar{F}(s) = F(s) - M(s)\,; \tag{47a}$$

$$K\bar{F}(s) = F(s) \ominus M(s)\,. \tag{47b}$$*

For the sake of brieferness we will consider only Eq. (47a) and assume that all the functions involved are impedances. The only purpose of M(s) is to obtain the H-class function multiplied with a positive constant K; the ER-class function F(s) and the H-class function $\bar{F}(s)$ are assumed to be of the same even rank 2ν.

For economical reasons we try to keep the rank of M(s) as low as possible. In the simplest event, M(s) = 1, realized by a unit resistance. We will discuss this case first in Sec. 3.1.

3.1 The H-Class Function $\bar{F}(s)$ Obtained by Subtracting a Constant K From F(s)

Assume a positive constant k < 1 that yields

$$K\bar{F}(s) = F(s) - k = K\bar{N}(s)/\bar{D}(s)\,, \tag{48}$$

where

$$\bar{N}(s) = [N(s) - kD(s)]/(1-k)\,, \tag{48a}$$

$$\bar{D}(s) = D(s)\,, \tag{48b}$$

$$K = 1-k\,. \tag{48c}$$

As in Eq. (16) let us similarly define

$$n = +\sqrt{D_o/N_o}\,. \tag{49}$$

and let

$$\bar{n} = n\sqrt{(1-k)/(1-n^2k)}\,. \tag{50}$$

Defining

$$n/n_o = nk + \sqrt{(1-k)(1-n^2k)}\,, \tag{51}$$

let

$$P(s)Q(s) = [n_o N(s) - D(s)]/(n_o - 1)\,. \tag{52}$$

---

*The notation a ⊖ b has the meaning of $ab/(a-b)$.

It can easily be shown that $\bar{d}(s)$, defined in (21), is

$$\bar{d}(s) = [N(s) - D(s)]/(\bar{n} - 1)(1 - k) . \tag{53}$$

But then Eq. (42) can be expressed as

$$\delta = - \sum_{i=0}^{\nu-1} c_i d_i = 0 , \tag{54}$$

where this time the coefficients c_i are derived from (52) instead of from (27), and $\bar{n}$ is derived from (50). Since

$$1/k = n_o^2 [n^2(1 - 2/n_o) + 1]/(n_o^2 - n^2) , \tag{55}$$

the constant k can accordingly be replaced by n_o. Instead of solving directly for k it is more convenient to first determine n_o. The H-class function $\bar{F}(s)$ in Eq. (48) has been found when, after a search with trial parameters n_o, Eq. (54) yields $\delta = 0$. We show this in Sec. 3.2.

3.2 Numerical Example No. 2 (Biquartic H-Class Function)

Let a biquartic ER-class impedance function be described by the coefficients listed in Table 8. The table also lists the coefficients of

$$d(s) = [N(s) - D(s)]/(n - 1). \tag{56}$$

The function F(s) is PR. The minimum of $\mathrm{Re}\,F(j\omega)$ is $F(0) = 1/n^2 = 0.4$. We disregard that F(s) - F(0) can be realized by the well-known ladder procedure. But we want to point out that any positive constant $0 \leq k < 0.4$ can be subtracted from F(s), and so the value n_o can range between 2.5 and $\sqrt{2.5} = 1.5811388$.

Let us first test whether F(s) is an H-class function. (This can be done either by subtracting the constant k = 0 or letting $n_o = \sqrt{2.5}$.) If we treat F(s) as though it is an H-class function, Eq. (55) will yield the solutions

$$s_4 = 1.2143093, \quad s_1 = 3.2212775, \quad s_2 = 4.7705955, \quad s_3 = 8.1341325. \tag{57}$$

With these solutions we obtain the results listed in Table 9. As the last column in Table 9 shows, F(s) is not an H-class function. We therefore use a series of trial parameters n_o to compute the δ_1, δ_2, δ_3 for the possible choices. Some of the results are presented in Table 10. As the table shows, choice No. 2 is the

Table 8

i	N_i	D_i	d_i
0	96.0	240	-247.7893405
1	133.4	195	-105.9987734
2	78.0	88	-17.2075930
3	15.6	16	-0.6883037
4	1.0	1	0

Table 9

Choice	P_o	P_1	Q_o	Q_1	δ
No. 1	-3.9116272	2.0069682	18.8046559	12.9047280	-336.1362632
No. 2	-5.7929785	3.5562862	26.2022980	11.3554100	-12.7247050
No. 3	-9.8773527	6.9198232	15.3674119	7.9918730	-201.7706416

Table 10

n_o	1.6	1.66	1.68	1.88	1.89
δ_1	-409.7952392	-476.4259531	-496.3981346	-653.3761346	-665.1590852
δ_2	-8.4908898	-0.5233455	+0.3875778	+0.3571147	-0.0203837
δ_3	-195.0626017	-175.4795503	-169.4798472	-120.4249847	-188.3943812

correct one. The value δ_2 changes its sign of polarity from n_o = 1.66 to 1.68 and again from 1.88 to 1.89. The exact zero crossings of δ_2 can be found by any well-known interpolation formula (we suggest the one by Aitken in Abramovitz and Stegun, 1964). The results of interpolation, together with the constants K, $\bar{n}$, and n, are presented in Table 11.

For K = 0.2 and k = 0.8 the function $\bar{F}(s)$ is the same as in Example No. 1. The impedance function has the same equivalent realizing circuits shown in Figure 4. But since we want to realize $K\bar{F}(s)$, the resistances, inductances, and inverse capacitances listed in Table 7 have to be multiplied by K = 0.2. Finally, to obtain the impedance F(s), both circuits have to be augmented with a series resistance R_o = k = 1-K = 0.8.

Table 11

n_o	15/9	1.8894704
δ_2	-0.0000013	+0.0000020
n	1.5811388	1.5811388
$\bar{n}$	2.0	3.6422085
K	0.2	0.3510830

With K = 0.3510830, the constants obtained for $\bar{F}(s)$ are:

$$\bar{n} = 3.6422085, \quad v = 0.0818905, \quad x = 0.3919552\,, \tag{58a}$$

and the coefficients

$$a_o = 6.0443381, \quad \alpha_o = 6.6029088, \quad b_o = 2.5409135, \quad \beta_o = 5.4181825. \tag{58b}$$

The impedance realizations of F(s) are again those shown in Figure 4, with the circuit elements listed in Table 7. The element values have accordingly to be multiplied by the factor K and the circuits have to be augmented with a series resistance of R_o = 1-K = 0.6489170 at the input to obtain realization of the impedance F(s). We have thus found four equivalent circuits.

We want to point out that if f(s) < 1, the constant k does not necessarily have to be positive. Since we have accepted one negative resistance in the realization, we can as well accept a second. There is also the chance that this second negative resistance may be cancelled out by a positive one if F(s) is part of a larger circuit.

3.3 The Function $\bar{F}(s)$ Obtained by Subtracting kf(s) From F(s)

If trials with an assumed function f(s) = 1 fail to yield an H-class function, we have to find a more costly function f(s) = n(s)/d(s). Considering the decomposition [Eq. (44)] only, let d(s) be a part of D(s),

$$D(s) = d(s)D'(s)\,. \tag{59}$$

Assume that the degree of n(s) is one order less than the degree of d(s). Then according to (44),

$$K\bar{F}(s) = (1-k)\bar{N}(s)/\bar{D}(s)\,, \tag{60}$$

with

$$\bar{N}(s) = [N(s) - kn(s)D'(s)]/(1-k); \tag{60a}$$

$$\bar{D}(s) = D(s) = d(s)D'(s) . \tag{60b}$$

We can also allow n(s) to be of the same degree as d(s), changing (60) and (60a) accordingly. We always have to ensure that $K\bar{F}(s)$ in (44) is PR, which is true when

$$\text{Re } F(j\omega) - \text{Re } kf(j\omega) \geq 0 . \tag{61}$$

Let us now consider the following example.

3.1 Numerical Example No. 3 (Bicubic H-Class Function)

For the coefficients of a bicubic impedance function F(s) listed in Table 12, we find that the denominator of F(s) is

$$D(s) = (s+s_o)(s^2 + D_1's + D_o') , \tag{62}$$

where

$$s_o = 0.5207709,\ D_1' = 0.4792291,\ D_o' = 3.6004314 . \tag{62a}$$

Table 12

i	N_i	D_i
0	1.74	1.875
1	0.192	3.85
2	7.95	1.0
3	1.0	1.0

We have chosen $f(s) = 1/(s+s_o)$. In Figure 5 we compare Re $F(j\omega)$ and Re $f(j\omega)$ versus $\Omega = \omega^2$. Since F(0) = 0.928, and f(0) = 1.9202301, the constant k must be less than 0.483. Factorization of P(s)Q(s) allows two choices, δ_1 and δ_2, whose values for a series of parameters k are listed in Table 13.

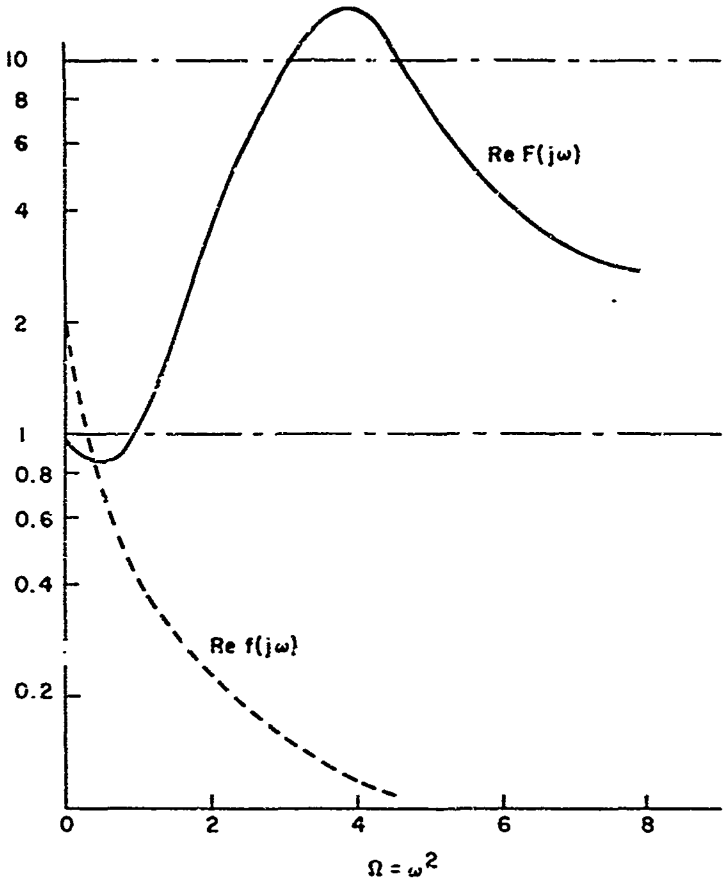

Figure 5. Example 2

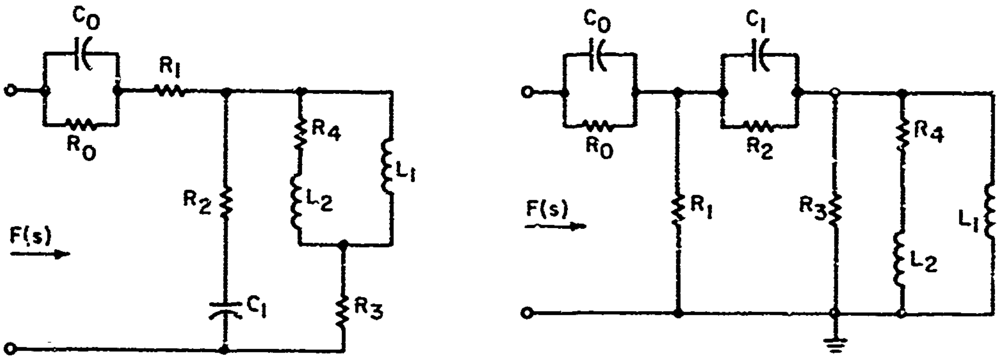

Figure 6. Two Equivalent Circuits Realizing the Driving-Point Impedance F(s), With (ν=3), of Example No. 3

Table 13

k	0.48	0.40	0.399	0.3995
δ_1	-0.2067047	-0.0007434	+0.0115423	+0.0053377
δ_2	-0.5809142	-4.0111988	-4.0452474	-4.0287219

Being satisfied with the small $\delta_1 = -0.0007434$ for $k = 0.40$ we go on to obtain

$$\bar{N}_o = 0.3, \quad \bar{N}_1 = 6.0, \quad \bar{N}_2 = 7.55, \quad \bar{N}_3 = 1.0 . \tag{63}$$

The constants for F(s) are

$$\bar{n} = 2.5, \quad v = 2.0, \quad x = 3.0 . \tag{64}$$

The coefficients of $\phi(s) = s(s+a_o)/(s+b_o)$ are

$$a_o = 0.8, \quad b_o = 0.5; \quad \Phi(s) = 1/s . \tag{65}$$

The two equivalent circuits realizing the driving-point impedance F(s) are shown in Figure 6, and the element values are listed in Table 14.

Table 14. Example No. 3

	T Circuit	π Circuit
R_o	0.7680920	0.7680920
R_1	0.24	0.24
R_2	0.36	0.16
R_3	-0.144	-0.4
R_4	2.304	6.4
L_1	1.728	4.8
L_2	2.88	8.0
C_o	2.5	2.5
C_1	1/4.05	1/1.8

I. COMMENTS

It must by now be apparent that the driving-point realizations discussed in this paper have some similarity to the well-known Brune realizations (1931). Our realization, however, is a single terminated two-port, whereas Brune's is a cascade of several two-ports. The single two-port allows us to induce the negative resistance. Where application of the Brune cycle requires that a duplex zero appear for the real component Re $F(j\omega)$, our method requires that either Eq. (42) or (54) yield $\delta = 0$. The Brune realization allows transforming the circuit into a Bott-Duffin (1949) circuit. We have not yet found a similar equivalent for our circuits even at the expense of using more elements to avoid the negative resistance. We can get a Bott-Duffin equivalent only when $\phi(s) = 1/\Phi(s)$; in such case, however, $F(s)$ is nothing more than a frequency-transformed biquadratic function. Although this event is almost trivial, our realization procedure at least offers the means of discovering the transformation. Whether an H-class function can always be derived from an ER-class function by splitting off a proper function $kf(s)$ will have to be determined empirically through a computer-aided design.

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13. ABSTRACT

A novel electrical circuit designed according to the theory postulated in this paper delivers a prescribed immittance response. A driving-point function $\overline{F}(s)$ is realized by a single-T circuit and/or a single-π circuit, implying a negative resistor but otherwise positive elements; a positive real immittance function F(s) is realized by the same circuits augmented by an immittance kf(s) of low rank at the input.

The circuit has high reliability and a rugged stability in the face of environmental changes. In addition, its low weight and small size fulfill requirements for military applications.

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